

Stability of the Multimegabit Telemetry Carrier Loop

D. W. Brown

TDA Technology Development

Basic sampled data loop stability is reviewed; the effect of an additional low-pass filter in the loop is analyzed. Resulting upper bounds on permissible loop bandwidth are established.

I. Introduction

The carrier tracking loop of the multimegabit telemetry demodulator currently under advanced systems development is of the second order sampled data type. In addition to the design point damping considerations typical of a continuous or analog loop, an additional potential instability exists for excessive gain (Ref. 1). Over and above that limitation, this intended implementation is confronted with the effects of a 10-kHz roll-off in the response of the Block IV VCO when operating in the long loop carrier reconstruction mode (Fig. 1 of Ref. 2). We will start with a simplified model (Fig. 2 of Ref. 1), add a low-pass time constant to the open loop transfer function, generalize the parameters, and develop a design guideline for this application.

II. The Model

Proceeding directly from Fig. 1, we can write the open loop transfer function in the variable z , where $z = e^{Ts}$:

$$OL(z) = \frac{\phi_o(z)}{\phi(z)} = AK_v F(z) \times Z \left\{ \frac{1 - e^{-Ts}}{(1 + T_x s)^2} \right\}$$

where $Z \{ \}$ denotes the z -transform equivalent to the bracketed Laplace transform. This yields

$$OL(z) = AK_v F(z) \left[\frac{Tz^{-1}}{1 - z^{-1}} - \frac{T_x(1 - e^{-T/T_x})z^{-1}}{1 - e^{-T/T_x}z^{-1}} \right]$$

For shorthand, let $\tau = T_x(1 - e^{-T/T_x})$:

$$OL(z) = AK_v \left(K_L + \frac{K_I}{1 - z^{-1}} \right) \left(\frac{Tz^{-1}}{1 - z^{-1}} - \frac{\tau z^{-1}}{1 - e^{-T/T_x}z^{-1}} \right) \quad (1)$$

III. The Continuous Equivalent

In order to generalize the parameters into more familiar terms, let us digress temporarily by applying the approximation

$$1 - z^{-1} = 1 - e^{-Ts} \approx Ts$$

for sampling rate $1/T$ large compared with frequencies of interest. Similarly,

$$z^{-1} \approx 1$$

For brevity we write the sample interval as simply T ; other multimegabit authors have used T_m or mT .

Then, for an ideal loop ($T_x = 0$), the open loop function reduces to

$$OL(s) \approx \frac{AK_v}{s} \left(K_L + \frac{K_I}{Ts} \right)$$

And the closed loop transfer function is

$$H(s) \approx \frac{\frac{AK_v}{s} \left(K_L + \frac{K_I}{Ts} \right)}{1 + \frac{AK_v}{s} \left(K_L + \frac{K_I}{Ts} \right)}$$

If we set

$$T_2 = T(K_L/K_I), \quad (2)$$

$$H(s) \approx \frac{1 + T_2 s}{1 + T_2 s + \frac{T}{AK_v K_I} s^2}$$

which is readily recognized as the continuous second-order loop, where

$$\omega_n^2 = \frac{AK_v K_I}{T} \text{ and } \zeta = \frac{\omega_n T_2}{2} \quad (3)$$

Hence

$$H(s) \approx \frac{1 + \frac{2\zeta}{\omega_n} s}{1 + \frac{2\zeta}{\omega_n} s + \frac{1}{\omega_n^2} s^2} \quad (4)$$

Provided the above approximation is maintained in design (expressed sometimes as $2\beta_L T \ll 1$), the continuous approximation and equivalences are useful in many aspects of performance analysis. But for the purposes of this study, it is preferred to retain the exact z-transform approach.

Before returning to the exact form, it should be noted that loop gain variations in A will occur due to AGC and limiter effects in the range of 10:1. Since

$$\frac{\zeta_{\max}^2}{\zeta_0^2} = \frac{A_{\max}}{A_0} \quad (5)$$

where subscript “0” denotes design point, and assuming $\zeta_0^2 = 1/2$,

$$\zeta_{\max}^2 \approx 5$$

To allow for implementation tolerances, the following analysis will consider

$$\frac{1}{4} \leq \zeta^2 \leq 8 \quad (6)$$

IV. Stability Analysis

Returning now to the sampled data transfer function (1) and substituting equivalences (2) and (3), we obtain

$$OL(z) = \frac{4\zeta^2}{T_2^2} \left(\frac{T + T_2 - T_2 z^{-1}}{1 - z^{-1}} \right) \left(\frac{Tz^{-1}}{1 - z^{-1}} - \frac{\tau z^{-1}}{1 - e^{-T/T_x} z^{-1}} \right) \quad (7)$$

Rearranging as a ratio of polynomials in z^{-1} ,

$$OL(z) = \frac{(T + T_2)(T - \tau)z^{-1} - \left[(T + T_2)(Te^{-T/T_x} - \tau) + T_2(T - \tau) \right] z^{-2} + T_2(Te^{-T/T_x} - \tau)z^{-3}}{\frac{T_2^2}{4\xi^2} \left[1 - (2 + e^{-T/T_x})z^{-1} + (1 + 2e^{-T/T_x})z^{-2} - e^{-T/T_x}z^{-3} \right]}$$

From

$$H(z) = \frac{OL(z)}{1 + OL(z)} = \frac{\text{NUM } OL(z)}{\text{DENOM } OL(z) + \text{NUM } OL(z)},$$

$$H(z) = \frac{(T + T_2)(T - \tau)z^{-1} - \left[(T + T_2)(Te^{-T/T_x} - \tau) + T_2(T - \tau) \right] z^{-2} + T_2(Te^{-T/T_x} - \tau)z^{-3}}{\frac{T_2^2}{4\xi^2} + \left[\frac{(T + T_2)(T - \tau)}{-\frac{T_2^2}{4\xi^2}(2 + e^{-T/T_x})} \right] z^{-1} - \left[\frac{(T + T_2)(Te^{-T/T_x} - \tau) + T_2(T - \tau)}{-\frac{T_2^2}{4\xi^2}(1 + 2e^{-T/T_x})} \right] z^{-2} + \left[\frac{T_2(Te^{-T/T_x} - \tau)}{-\frac{T_2^2}{4\xi^2}e^{-T/T_x}} \right] z^{-3}} \quad (8)$$

where $\tau = T_x(1 - e^{-T/T_x})$, as previously defined.

As a partial check on results, set $\tau = T_x = 0$ for the ideal case:

$$H(z) = \frac{(T + T_2)Tz^{-1} - TT_2z^{-2}}{\left[\frac{T_2^2}{4\xi^2} \right] - \left[2\frac{T_2^2}{4\xi^2} - (T + T_2)T \right] z^{-1} + \left[\frac{T_2^2}{4\xi^2} - TT_2 \right] z^{-2}} \quad (9)$$

It turns out that the principal stability criterion employed in Ref. 1 is equivalent to requiring that the sum of the bracketed coefficients in the denominator exceed zero, or

$$\left(\frac{T}{2} + T_2 \right) T < 2 \frac{T_2^2}{4\xi^2}$$

And that the gain margin may be computed as the ratio by which it does, such that

$$GM = \frac{2 \frac{T_2^2}{4\xi^2}}{\left(\frac{T}{2} + T_2 \right) T} = \frac{1}{2\xi^2} \frac{T_2}{T} \frac{1}{\left(1 + \frac{T}{2T_2} \right)} \quad (10)$$

But since $T/T_2 = K_I/K_L$, by equation (2), and K_I/K_L is to be implemented on the order of 2^{-8} (another manifestation of the $2\beta_L T \ll 1$ criterion),

$$GM \approx \frac{1}{2\xi^2} \frac{K_L}{K_I} = \frac{2^7}{\xi^2}$$

yielding:

$$GM \left(\xi^2 = \frac{1}{2} \right) = 2^8 = 256 \text{ or } 48 \text{ dB at design point}$$

and

$$GM(\xi^2 = 8) = 2^4 = 16 \text{ or } 24 \text{ dB at maximum gain}$$

Holmes' result of 35.2 dB was probably based on a $\xi^2 = 1/4$ design point and $A_{\max}/A_0 \approx 8.9$, yielding $\xi_{\max}^2 \approx 2.225$ and $GM = 2^7/2.225 \approx 57.5$ or 35.2 dB.

While it is necessary to design with an adequate gain margin, it is also required in the "real world" of equation (1) to maintain the "extra" time constant T_x smaller than the reciprocal frequencies of interest. This situation can be loosely related as

$$\frac{1}{\omega_n} > T_x > T \quad (11)$$

A rule of thumb is recalled from continuous loop design:

$$\omega_n T_x < 0.05 \quad (12)$$

But this is too empirical; we need to calibrate the effect of T_x on loop margins or transient performance in order to specify the design relationships with more rigor. And although it might be possible to apply numerical stability criteria to equation (8) as we did above for the ideal case, the approach taken is to perform the inversion of the z -transform pulse response function into the time domain and parametrically evaluate the overshoot of the response to a step input. And, rather than attempt a formal inversion, an alternate technique is employed (Ref. 3, p. 57). If the desired pulse response function is expanded into a power series in z^{-n} by the process of long division, then the coefficients represent the magnitudes of the time response at times $t = nT$. The mechanics are handled as follows:

- (1) For each parametric set of variables, T , T_2 , T_x , and ζ^2 , compute the coefficients for equation (8).
- (2) Multiply $H(z)$ by $\phi_i(z) = 1/(1 - z^{-1})$ to obtain $\phi_o(z)$ for a step input and regroup the denominator in z^{-n} series.
- (3) Enter the coefficients of $\phi_o(z)$ numerator and denominator in a desk calculator programmed to compute the first 38 coefficients of the power series.
- (4) Plot these coefficients versus $t = nT$ to obtain a sampled version of $\phi_o(t)$.

Returning to the problem at hand, it is likely that for a given sample rate T , all parameters of Fig. 1 would remain fixed with the exception of A , which will vary with input signal SNR as noted above. Intuitively, then, a given constraining value of T_x is most likely to have the greatest impact for the widest bandwidth, i.e., strong signal. Take, then, as a starting point $\zeta^2 = 8$. Arbitrarily, set $T_2 = 16$ as a convenient scaling within the $38T$ -second window offered by the calculator program.

As an initial value for T_x use (12):

$$\omega_n T_x = 2\zeta \frac{T_x}{T_2} < 0.05$$

or

$$T_x < 0.025 \frac{T_2}{\zeta} = \frac{\sqrt{2}}{10} \approx \frac{1}{8}$$

What, then, in our scale model loop, should we use for T ? A true scaling would set

$$T = T_2 \frac{K_I}{K_L} = 16 \times 2^{-8} = \frac{1}{16}$$

This would provide us only with $t = nT = 38/16$ -second visibility due to the limited range of the program. Let us reduce the gain margin of our scale model slightly and let $T = 1/10$, providing $t = 38/10 \approx 4$ seconds.

Taking, now, as a baseline $T_2 = 16$, $\zeta^2 = 8$, $T = 1/10$, and $T_x = 0$, we obtain

$$\phi_o(z) = \frac{1.61z^{-1} - 1.60z^{-2}}{8 - 22.39z^{-1} + 20.79z^{-2} - 6.40z^{-3}}$$

and for $T_x = 1/8$,

$$\phi_o(z) = \frac{0.5018z^{-1} - 11.39z^{-2} - 0.3824z^{-3}}{8 - 27.0926z^{-1} + 34.1683z^{-2} - 19.0525z^{-3} + 3.9768z^{-4}}$$

The resulting time series are plotted in Fig. 2. We note that the "rule of thumb" value of $\omega_n T_x$ results in a discernable, but inconsequential, change in transient response. Recalling that the overshoot of the ideal loop with $\zeta^2 = 1/2$ design point damping is approximately 20 percent, let us keep increasing T_x parametrically until some arbitrary upper bound is reached. A value of 40 percent is taken as a likely upper limit of desirable response. Values of $T_x = 1/4, 1/2, 1$ are seen in Fig. 2 to cover this range.

In order to evaluate the sensitivity of the arbitrary reduction in gain margin for this scale model, we can plot peak overshoot (evaluated as above) versus T .

From Fig. 3 we can conclude that for $T = 1/10$ or even $1/8$, the model closely approximates the actual design at a gain margin of 24 dB.*

*Strictly speaking, gain margin as defined in (10) is not applicable in the presence of T_x ; however, it remains a useful, well-defined characteristic.

Now that a tentative worse case $T_x/T_2 = 1/16$ has been established, it would be useful to vary ξ^2 as a parameter, thus validating $\xi^2 = 8$ as the maximum overshoot condition. Figure 4 presents the full range of ξ^2 in octave steps for both $T_x/T_2 = 1/16$ and $T_x = 0$. In order to fit the region of interest in $t = nT$ as ξ^2 is varied, several different values of T were employed, always keeping the modelled gain margin of 18 dB or better (see Fig. 3), thus maintaining reasonable accuracy.

V. Conclusions

We have seen that the multimegabit sampled data loop may be closely approximated as a continuous loop and that the gain margin may be readily computed as

$$GM = \frac{1}{2\xi^2} \frac{T_2}{T} = \frac{1}{2\xi^2} \frac{K_L}{K_I}$$

whatever the detailed design may be. Additionally, Figs. 2 and 4 offer a basis for specifying maximum tolerable extra time constant or, conversely, the maximum tolerable nominal bandwidth for a given T_x .

For example, from the figures, one could specify that for $8 \geq \xi^2 \geq 1/2$, and overshoot comparable to the ideal at $\xi^2 = 1/2$:

$$\frac{T_x}{T_2} \leq \frac{1}{32} \quad \text{or} \quad T_x \leq \frac{T}{32} \frac{K_L}{K_I}$$

or

$$2\beta_{L_0} T_x \leq \frac{3}{64} \quad \text{at} \quad \xi_0^2 = \frac{1}{2}$$

Alternatively, for $8 \geq \xi^2 \geq 1/4$, and overshoot less than 40 percent,

$$\frac{T_x}{T_2} \leq \frac{1}{16}$$

for a $2\beta_{L_0} T_x$ product $\leq 3/32$ at $\xi_0^2 = 1/2$. In any case, this approach offers a relatively exact analysis and perspective on the compound effects of gain margin and extra time constant.

References

1. Holmes, J. K., "Stability Analysis of the Multimegabit Telemetry Demodulator/Detector Design," *DSN Progress Report 42-51*, pp. 31-38, Jet Propulsion Laboratory, Pasadena, Calif., June 15, 1979.
2. Woo, K. T., "Effects of Asymmetric Passband Filtering on the Phase of the Costas Loop's Reconstructed Carrier," *DSN Progress Report 42-51*, pp. 105-112, Jet Propulsion Laboratory, Pasadena, Calif., June 15, 1979.
3. Ragazzini, J. R., and Franklin, G. F., "Sampled-Data Control Systems," McGraw-Hill, New York, 1958.

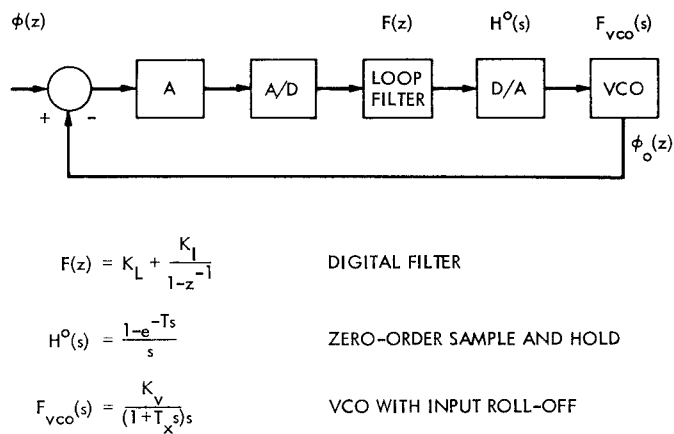


Fig. 1. Simplified loop model with VCO roll-off

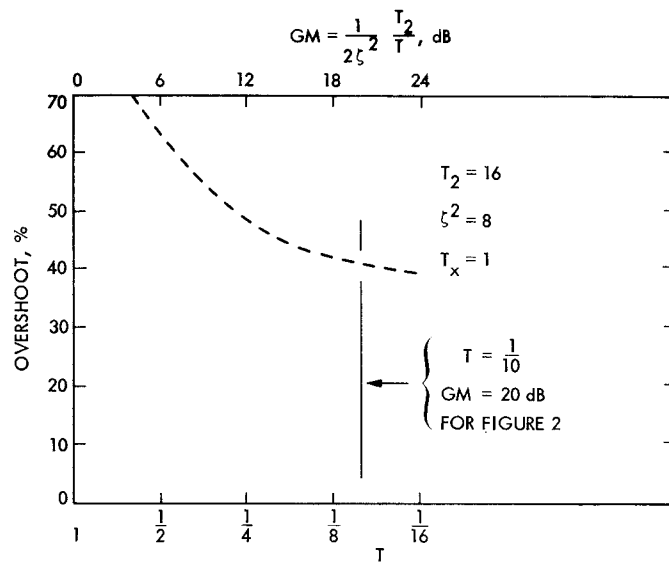


Fig. 3. Sensitivity of overshoot to gain margin

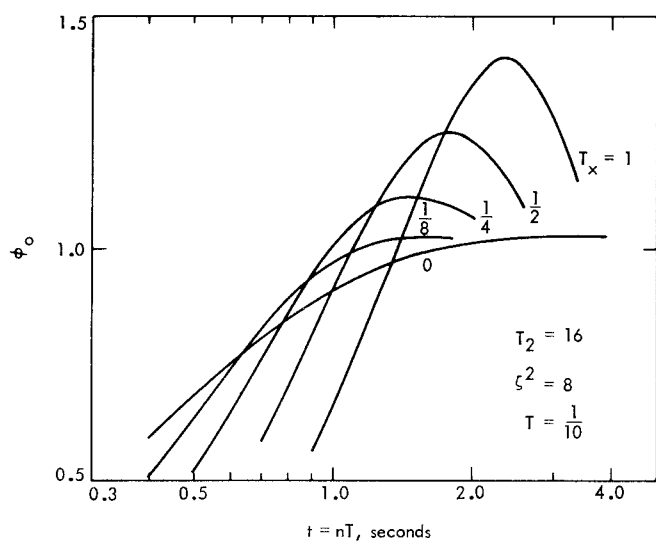


Fig. 2. Transient response vs extra time constant

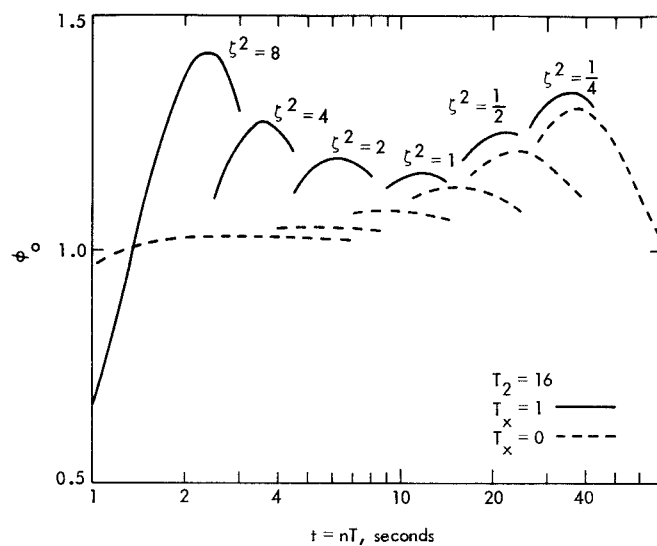


Fig. 4. Transient response vs damping